The Degree of an Eight-Dimensional Real Quadratic Division Algebra is 1, 3, or 5

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Abstract

A celebrated theorem of Hopf, Bott, Milnor, and Kervaire [11],[1],[12] states that every finite-dimensional real division algebra has dimension 1, 2, 4, or 8. While the real division algebras of dimension 1 or 2 and the real quadratic division algebras of dimension 4 have been classified [6],[3],[9], the problem of classifying all 8-dimensional real quadratic division algebras is still open. We contribute to a solution of that problem by proving that every 8-dimensional real quadratic division algebra has degree 1, 3, or 5. This statement is sharp. It was conjectured in [7].

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1 Introduction

Let A be an algebra over a field k, i.e. a vector space over k equipped with a k-bilinear multiplication $A \times A \to A$, $(x, y) \mapsto xy$. Every element $a \in A$ determines k-linear operators $L_a : A \to A$, $x \mapsto ax$ and $R_a : A \to A$, $x \mapsto xa$. A *division algebra* over k is a non-zero k-algebra A such that L_a and R_a are bijective for all $a \in A \setminus \{0\}$. A *quadratic algebra* over k is a non-zero k-algebra A with unity 1, such that for all $x \in A$ the sequence $1, x, x^2$ is k-linearly dependent.

Here we focus on quadratic division algebras which are real (i.e. $k = \mathbb{R}$) and finite-dimensional. They form a category $\mathscr{D}^q$ whose morphisms $f : A \to B$ are non-zero linear maps satisfying $f(xy) = f(x)f(y)$ for all $x, y \in A$. For any positive integer n, the class of all n-dimensional objects in $\mathscr{D}^q$ forms a full subcategory $\mathscr{D}^q_n$ of $\mathscr{D}^q$. The (1,2,4,8)-theorem for finite-dimensional real division algebras implies that $\mathscr{D}^q = \mathscr{D}^q_1 \cup \mathscr{D}^q_2 \cup \mathscr{D}^q_4 \cup \mathscr{D}^q_8$. It is well-known (see e.g. [9]) that both $\mathscr{D}^q_1$ and $\mathscr{D}^q_2$ consist of one isoclass only, represented by $\mathbb{R}$ and $\mathbb{C}$ respectively. The category $\mathscr{D}^q_4$ is no longer trivial, but still its structure is well-understood and its objects are classified by a 9-parameter family [3],[5],[9]. In contrast, the category $\mathscr{D}^q_8$ seems

to be much more difficult to approach. The problem of understanding its structure appears to have a very hard core, which still is far from a finishing solution. Among the insight so far obtained, the *degree* of an 8-dimensional real quadratic division algebra is notable. Following [7] it is a natural number $\deg(A)$ associated with any $A \in \mathscr{D}_8^q$, which is invariant under isomorphisms and satisfies the estimate $1 \leq \deg(A) \leq 5$. Examples of algebras $A \in \mathscr{D}_8^q$ having degree 1, 3, and 5 respectively are also given in [7]. For every $d \in \{1, \ldots, 5\}$, the class of all algebras in $\mathscr{D}_8^q$ having degree d forms a full subcategory $\mathscr{D}_8^{qd}$ of $\mathscr{D}_8^q$.

In the present article we resume the investigation of $\deg(A)$. In section 2 we apply topological arguments to prove that $\deg(A)$ always is odd. It follows that the category $\mathscr{D}_8^q$ decomposes into three non-empty blocks $\mathscr{D}_8^{q1}, \mathscr{D}_8^{q3}$, and $\mathscr{D}_8^{q5}$. In section 3 we summarize our structural insight into these three blocks, concluding that the problem of understanding the structure of $\mathscr{D}_8^q$ has been reduced to the problem of understanding the structure of the category of all 7-dimensional *dissident algebras* having degree 3 or 5.

2 Main result

Towards our main result we need to recall a few facts related to the notion of a dissident map. Let V be a vector space over a field k. Following [8], a *dissident map* on V is a k-linear map $\eta : V \wedge V \to V$ such that $v, w, \eta(v \wedge w)$ are k-linearly independent whenever v, w are. It is well-known that a finite-dimensional real vector space V admits a dissident map if and only if $\dim(V) \in \{0, 1, 3, 7\}$ [4, Proposition 7].

Let $\eta : \mathbb{R}^7 \wedge \mathbb{R}^7 \to \mathbb{R}^7$ be a dissident map. We equip $\mathbb{R}^7$ with the standard scalar product $\mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}$, $(v, w) \mapsto v^t w$. For every $v \in \mathbb{R}^7 \setminus \{0\}$, the subspace $\eta(v \wedge v^\perp) = \{\eta(v \wedge w) \mid w \in v^\perp\}$ in $\mathbb{R}^7$ is a hyperplane, which only depends on the line $[v]$ spanned by v. Thus η induces a map

$$\eta_{\mathbb{P}} : \mathbb{P}(\mathbb{R}^7) \to \mathbb{P}(\mathbb{R}^7),\ \eta_{\mathbb{P}}([v]) = \Big(\eta(v \wedge v^\perp)\Big)^\perp,$$

which actually is bijective [8, Proposition 2.2]. Following [7], a *lifting* of $\eta_{\mathbb{P}}$ is a map $\Phi : \mathbb{R}^7 \to \mathbb{R}^7$, $\Phi(v) = (\varphi_1(v), \ldots, \varphi_7(v))$, satisfying the following three conditions: (a) all component maps $\varphi_1, \ldots, \varphi_7$ are homogeneous real polynomials of common degree $d \geq 1$; (b) if $v \in \mathbb{R}^7 \setminus \{0\}$, then $\Phi(v) \neq 0$ and $[\Phi(v)] = \eta_{\mathbb{P}}([v])$; (c) the polynomials $\varphi_1, \ldots, \varphi_7$ are relatively prime.

According to [7, Theorem 2.4] a lifting Φ of $\eta_{\mathbb{P}}$ exists, is unique up to non-zero real multiples, and satisfies $1 \leq d \leq 5$. It is therefore justified to define the *degree* of a dissident map η on $\mathbb{R}^7$ by $\deg(\eta) := \deg(\Phi) := d$.

Proposition 2.1. *The degree of any dissident map on $\mathbb{R}^7$ is odd.*

Proof. Our proof of Proposition 2.1 is topological in nature and, in particular, uses the fundamental group of the projective space $\mathbb{P}(\mathbb{R}^7)$.

Let $\pi : \mathbb{R}^7 \setminus \{0\} \to \mathbb{P}(\mathbb{R}^7)$ be the quotient map mapping a vector $v \in \mathbb{R}^7 \setminus \{0\}$ to the line $[v] \in \mathbb{P}(\mathbb{R}^7)$ spanned by v. We equip $\mathbb{R}^7 \setminus \{0\}$ with its standard Euclidean topology and $\mathbb{P}(\mathbb{R}^7)$ with the quotient topology. For a point $z \in \mathbb{P}(\mathbb{R}^7)$ let $\pi_1(\mathbb{P}(\mathbb{R}^7), z)$ be the fundamental group of $\mathbb{P}(\mathbb{R}^7)$ based at z. We recall that

$$\pi_1(\mathbb{P}(\mathbb{R}^7), z) = \mathbb{Z}_2 \ ,$$

$\mathbb{Z}_2 = \{0, 1\}$ being the group of integers modulo 2 [2, Section III.5].

The map $\pi : \mathbb{R}^7 \setminus \{0\} \to \mathbb{P}(\mathbb{R}^7)$ is a locally trivial fibration with the fibre homeomorphic to $\mathbb{R} \setminus \{0\}$. We choose a point $z \in \mathbb{P}(\mathbb{R}^7)$ and a vector $\tilde{z} \in \mathbb{R}^7 \setminus \{0\}$ such that $\pi(\tilde{z}) = z$. Then every vector w belonging to the fibre $\pi^{-1}(z)$ is of the form $w = q\tilde{z}$ with $q \in \mathbb{R},\, q \neq 0.$

Denote by I the unit interval $[0, 1]$.

Let $\sigma : I \to \mathbb{P}(\mathbb{R}^7)$ be a loop in $\mathbb{P}(\mathbb{R}^7)$ at the point z i.e. σ is a continuous mapping such that $\sigma(0) = \sigma(1) = z$. Since $\pi : \mathbb{R}^7 \setminus \{0\} \to \mathbb{P}(\mathbb{R}^7)$ is a fibration, it follows from the Homotopy Lifting Theorem, [2, Theorem VII.6.4], that the loop $\sigma : I \to \mathbb{P}(\mathbb{R}^7)$ can be lifted to a continuous mapping $\tilde{\sigma} : I \to \mathbb{R}^7 \setminus \{0\}$ such that $\pi \circ \tilde{\sigma} = \sigma$ and $\tilde{\sigma}(0) = \tilde{z}$. (Observe that, in general, $\tilde{\sigma}$ will not be a loop but just a path.) Since $\sigma(1) = z$, it follows that $\tilde{\sigma}(1) \in \pi^{-1}(z)$ and, hence, that $\tilde{\sigma}(1) = q\tilde{z}$ for some $q = q(\sigma) \in \mathbb{R},\, q \neq 0.$ (Actually, the real number q depends not only on the loop σ but also on the choice of the lifting $\tilde{\sigma}$ which is not unique.) The next lemma is rather obvious. We include a proof for convenience of the reader.

Lemma 2.2. *The loop $\sigma : I \to \mathbb{P}(\mathbb{R}^7)$ represents the trivial element in $\pi_1(\mathbb{P}(\mathbb{R}^7), z) = \mathbb{Z}_2$ if and only if $q > 0$.*

Proof. ($\Leftarrow$) Suppose that $q > 0$. Then there is a line segment $\tilde{\tau}$ in $\pi^{-1}(z)$ from $q\tilde{z}$ to $\tilde{z}$ consisting of points of the form $s\tilde{z}$ with s between 1 and q. The path product $\overline{\sigma} = \tilde{\sigma} * \tilde{\tau}$ is now a loop in $\mathbb{R}^7 \setminus \{0\}$ starting and ending at $\tilde{z}$. The space $\mathbb{R}^7 \setminus \{0\}$ is homotopy equivalent to the 6-dimensional sphere S^6 and hence the fundamental group $\pi_1(\mathbb{R}^7 \setminus \{0\}, \tilde{z}) = 0$ is trivial. Therefore the loop $\overline{\sigma} = \tilde{\sigma} * \tilde{\tau}$ is null-homotopic in $\mathbb{R}^7 \setminus \{0\}$. It follows that the loop $\pi \circ \overline{\sigma} = (\pi \circ \tilde{\sigma}) * (\pi \circ \tilde{\tau}) = \sigma * (\pi \circ \tilde{\tau})$ is null-homotopic in $\mathbb{P}(\mathbb{R}^7)$. Since $\pi \circ \tilde{\tau}$ is a constant loop, the loops $\sigma * (\pi \circ \tilde{\tau})$ and σ are homotopic. Therefore the loop σ is null-homotopic in $\mathbb{P}(\mathbb{R}^7)$ and represents the trivial element in $\pi_1(\mathbb{P}(\mathbb{R}^7), z)$.

($\Rightarrow$) Suppose that the loop $\sigma : I \to \mathbb{P}(\mathbb{R}^7)$ represents the trivial element in $\pi_1(\mathbb{P}(\mathbb{R}^7), z)$. Thus there exists a continuous mapping (homotopy) $H : I \times I \to \mathbb{P}(\mathbb{R}^7)$ such that $H(t, 0) = \sigma(t)$ and $H(0, s) = H(1, s) = H(t, 1) = z$ for all $t, s \in I$. Again, according to the Homotopy Lifting Theorem there exists a continuous mapping (lift) $\tilde{H} : I \times I \to \mathbb{R}^7 \setminus \{0\}$ such that $\pi \circ \tilde{H} = H$ and $\tilde{H}(t, 0) = \tilde{\sigma}(t)$, $\tilde{H}(0, s) = \tilde{z}$ and $\tilde{H}(1, s) = \tilde{\sigma}(1)$ for all $t, s \in I$. It follows, in particular, that $\tilde{H}(t, 1) \in \pi^{-1}(z)$ for all $t \in I$ and that $\tilde{H}(0, 1) = \tilde{z}$ while $\tilde{H}(1, 1) = \tilde{\sigma}(1) = q\tilde{z}$. In other words, $\tilde{H}(t, 1)$, $t \in I$,

is an arc in $\pi^{-1}(z)$ from $\tilde{z}$ to $\tilde{\sigma}(1) = q\tilde{z}$. As $\pi^{-1}(z)$ consists of points of the form $r\tilde{z}$ with $r \in \mathbb{R} \setminus \{0\}$ it follows that $q > 0$.

That completes the proof of Lemma 2.1. $\square$

The diagram

$$\begin{array}{ccc} \mathbb{R}^7 \setminus \{0\} & \xrightarrow{\Phi} & \mathbb{R}^7 \setminus \{0\} \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ \mathbb{P}(\mathbb{R}^7) & \xrightarrow{\eta_{\mathbb{P}}} & \mathbb{P}(\mathbb{R}^7) \end{array}$$

commutes. Since the components $\varphi_1, ..., \varphi_7$ of Φ are polynomials, the map $\Phi : \mathbb{R}^7 \setminus \{0\} \to \mathbb{R}^7 \setminus \{0\}$ is continuous. It follows that the map $\eta_{\mathbb{P}} : \mathbb{P}(\mathbb{R}^7) \to \mathbb{P}(\mathbb{R}^7)$ is also continuous. The map $\eta_{\mathbb{P}}$ is bijective [8, Proposition 2.2] and the projective space $\mathbb{P}(\mathbb{R}^7)$ is compact and Hausdorff. Therefore $\eta_{\mathbb{P}}$ is a homeomorphism. Let us choose a point $z_0 \in \mathbb{P}(\mathbb{R}^7)$ and let us denote by z_1 the point $\eta_{\mathbb{P}}(z_0) \in \mathbb{P}(\mathbb{R}^7)$. The homeomorphism $\eta_{\mathbb{P}}$ induces a group isomorphism

$$\eta_{\mathbb{P}*} : \pi_1(\mathbb{P}(\mathbb{R}^7), z_0) \xrightarrow{\cong} \pi_1(\mathbb{P}(\mathbb{R}^7), z_1) \ .$$

Let us choose a point $\tilde{z}_0 \in \pi^{-1}(z_0) \subset \mathbb{R}^7 \setminus \{0\}$. Let us denote by $\tilde{z}_1$ the point $\Phi(\tilde{z}_0) \in \mathbb{R}^7 \setminus \{0\}$. Then $\tilde{z}_1 = \Phi(\tilde{z}_0) \in \pi^{-1}(z_1)$.

Since the space $\mathbb{R}^7 \setminus \{0\}$ is path-connected, we can find a path $\tilde{\alpha} : I \to \mathbb{R}^7 \setminus \{0\}$ such that $\tilde{\alpha}(0) = \tilde{z}_0$ and $\tilde{\alpha}(1) = -\tilde{z}_0$. Then the composition $\alpha = \pi \circ \tilde{\alpha} : I \to \mathbb{P}(\mathbb{R}^7)$ is a loop in $\mathbb{P}(\mathbb{R}^7)$ which starts and ends at $z_0 = \pi(\tilde{z}_0) = \pi(-\tilde{z}_0)$. The path $\tilde{\alpha}$ is a lift of the loop α to $\mathbb{R}^7 \setminus \{0\}$ which starts at $\tilde{z}_0$ and ends at $-\tilde{z}_0$. Thus, according to Lemma 2.1, the loop α represents the nontrivial element of $\pi_1(\mathbb{P}(\mathbb{R}^7), z_0)$.

Let us now consider the path $\tilde{\beta} = \Phi \circ \tilde{\alpha} : I \to \mathbb{R}^7 \setminus \{0\}$ in $\mathbb{R}^7 \setminus \{0\}$ starting at $\tilde{z}_1 = \Phi(\tilde{z}_0) \in \pi^{-1}(z_1)$ and ending at $\tilde{z}_2 = \Phi(-\tilde{z}_0) \in \pi^{-1}(z_1)$. If $d = \deg(\Phi)$ is the degree of Φ then, by the definition of the degree, $\tilde{z}_2 = \Phi(-\tilde{z}_0) = (-1)^d\Phi(\tilde{z}_0) = (-1)^d\tilde{z}_1$.

The composition $\beta = \pi \circ \tilde{\beta} : I \to \mathbb{P}(\mathbb{R}^7)$ is a loop in $\mathbb{P}(\mathbb{R}^7)$ which starts and ends at the point $z_1 = \pi(\tilde{z}_1) = \pi(\tilde{z}_2)$. Moreover,

$$\beta = \pi \circ \tilde{\beta} = \pi \circ \Phi \circ \tilde{\alpha} = \eta_{\mathbb{P}} \circ \pi \circ \tilde{\alpha} = \eta_{\mathbb{P}} \circ \alpha \ .$$

Thus the homotopy class $[\beta]$ of the loop β in $\pi_1(\mathbb{P}(\mathbb{R}^7), z_1)$ is equal to $[\eta_{\mathbb{P}} \circ \alpha] = \eta_{\mathbb{P}*}[\alpha]$. As the homotopy class $[\alpha]$ of α in $\pi_1(\mathbb{P}(\mathbb{R}^7), z_0)$ was non-trivial and $\eta_{\mathbb{P}*} : \pi_1(\mathbb{P}(\mathbb{R}^7), z_0) \to \pi_1(\mathbb{P}(\mathbb{R}^7), z_1)$ was an isomorphism, it follows that β represents the non-trivial element of $\pi_1(\mathbb{P}(\mathbb{R}^7), z_1)$.

On the other hand the path $\tilde{\beta} : I \to \mathbb{R}^7 \setminus \{0\}$ is a lifting of the loop β to $\mathbb{R}^7 \setminus \{0\}$ which starts at the point $\tilde{z}_1 \in \mathbb{R}^7 \setminus \{0\}$ and ends at the point $\tilde{z}_2 = (-1)^d\tilde{z}_1$. Since β represents the non-trivial element of $\pi_1(\mathbb{P}(\mathbb{R}^7), z_1)$, it follows from Lemma 2.1 that $(-1)^d < 0$ and, thus, the degree $d = \deg(\Phi)$ is odd.

That completes the proof of Proposition 2.1. $\square$

Remark 2.3. *The space* $\mathbb{R}^7 \setminus \{0\}$ *is homotopy equivalent to the* 6*-dimensional sphere* S^6. *Its homology group* $H_6(\mathbb{R}^7 \setminus \{0\}, \mathbb{Z})$ *with coefficients in* $\mathbb{Z}$ *in dimension* 6 *is isomorphic to the group of integers,* $H_6(\mathbb{R}^7 \setminus \{0\}, \mathbb{Z}) \cong \mathbb{Z}$. *The continuous mapping* $\Phi : \mathbb{R}^7 \setminus \{0\} \to \mathbb{R}^7 \setminus \{0\}$ *induces a group homomorphism* $\Phi_* : H_6(\mathbb{R}^7 \setminus \{0\}, \mathbb{Z}) \to H_6(\mathbb{R}^7 \setminus \{0\}, \mathbb{Z})$ *which is given by multiplication by an integer usually called the (topological) degree of the map* Φ. *Since the map* $\eta_{\mathbb{P}} : \mathbb{P}(\mathbb{R}^7) \to \mathbb{P}(\mathbb{R}^7)$ *is a homeomorphism and the algebraic degree* $d = deg(\Phi)$ *in the sense of this paper is odd, it is rather easy to see that also the mapping* $\Phi : \mathbb{R}^7 \setminus \{0\} \to \mathbb{R}^7 \setminus \{0\}$ *is a homeomorphism. It follows that* $\Phi_* : H_6(\mathbb{R}^7 \setminus \{0\}, \mathbb{Z}) \to H_6(\mathbb{R}^7 \setminus \{0\}, \mathbb{Z})$ *is an isomorphism and, hence, its topological degree can only be equal to* ± 1. *Thus the topological degree of the map* Φ *and its algebraic degree* $d = deg(\Phi)$ *in the sense of* [7] *and of the present paper are different notions, at least when the algebraic degree* $d = deg(\Phi)$ *is equal to* 3 *or* 5.

Now let A be an 8-dimensional real quadratic division algebra. Since A is a quadratic algebra, Frobenius's lemma [10],[13] applies. It asserts that the set $V = \{v \in A \setminus \mathbb{R}1 \mid v^2 \in \mathbb{R}1\} \cup \{0\}$ of all purely imaginary elements in A is a hyperplane in A, such that $A = \mathbb{R}1 \oplus V$. This Frobenius decomposition of A gives rise to the $\mathbb{R}$-linear maps $\varrho : A \to \mathbb{R}$ and $\iota : A \to V$ such that $a = \varrho(a)1 + \iota(a)$ for all $a \in A$. The induced algebra structure on V, i.e. the bilinear map $\eta : V \times V \to V$, $\eta(v, w) = \iota(vw)$, is anticommutative. Therefore it may be identified with the linear map $\eta : V \wedge V \to V$, $\eta(v \wedge w) = \iota(vw)$. Since A is a division algebra, this linear map η is dissident [14]. Any choice of a basis in V identifies η with a dissident map η on $\mathbb{R}^7$, and the degree of η does not depend on the chosen basis. It is therefore justified to define the *degree* of an 8-dimensional real quadratic division algebra A by $\deg(A) := \deg(\eta)$.

Corollary 2.4. *The degree of any 8-dimensional real quadratic division algebra is 1, 3, or 5.*

Proof. Let $A \in \mathscr{D}_8^q$. Then $\deg(A) = \deg(\eta) = d$, where $1 \le d \le 5$ by [7, Theorem 2.4], and $\deg(\eta)$ is odd by Proposition 2.1. □

Corollary 2.5. *The category* $\mathscr{D}_8^q$ *decomposes into its non-empty full subcategories* $\mathscr{D}_8^{q1}, \mathscr{D}_8^{q3}$, *and* $\mathscr{D}_8^{q5}$.

Proof. Corollary 2.4 states that the object class of $\mathscr{D}_8^q$ is the disjoint union of the object classes of $\mathscr{D}_8^{q1}$, $\mathscr{D}_8^{q3}$, and $\mathscr{D}_8^{q5}$.

Let $f : A \to A'$ be a morphism in $\mathscr{D}_8^q$. The algebra structures on A and A' induce dissident maps η and η' on the purely imaginary hyperplanes V and V' of A and A' respectively. Now f is injective because A is a division algebra, and furthermore even bijective because $\dim(A) = \dim(A')$ is finite. So f is an isomorphism of algebras. It induces an isomorphism of dissident maps $\sigma : \eta \to \eta'$, i.e. a linear bijection $\sigma : V \to V'$ satisfying $\sigma\eta(v \wedge w) =$

$\eta'(\sigma(v) \wedge \sigma(w))$ for all $v, w \in V$. We conclude with [7, Proposition 3.1] that $\deg(\eta) = \deg(\eta')$, hence $\deg(A) = \deg(A')$. Thus f is a morphism in $\mathscr{D}_8^{qd}$ for some $d \in \{1, 3, 5\}$.

Altogether this proves the decomposition $\mathscr{D}_8^q = \mathscr{D}_8^{q1} \amalg \mathscr{D}_8^{q3} \amalg \mathscr{D}_8^{q5}$ of the category $\mathscr{D}_8^q$. Non-emptyness of its blocks $\mathscr{D}_8^{q1}, \mathscr{D}_8^{q3}$, and $\mathscr{D}_8^{q5}$ follows from [7, Section 6], where objects are constructed for each of them. □

3 On the structure of $\mathscr{D}_8^{q1}$, $\mathscr{D}_8^{q3}$, and $\mathscr{D}_8^{q5}$

Corollary 2.5 reduces the problem of understanding the structure of $\mathscr{D}_8^q$ to the problem of understanding the structures of $\mathscr{D}_8^{q1}, \mathscr{D}_8^{q3}$, and $\mathscr{D}_8^{q5}$. We proceed to summarize the present state of knowledge regarding the latter problem.

A *dissident triple* (V, ξ, η) consists of a (finite-dimensional) Euclidean space $V = (V, \langle \cdot, \cdot \rangle)$, a linear form $\xi : V \wedge V \to \mathbb{R}$, and a dissident map $\eta : V \wedge V \to V$. The class of all dissident triples forms a category $\mathscr{V}$ whose morphisms $\varphi : (V, \xi, \eta) \to (V', \xi', \eta')$ are orthogonal linear maps $\varphi : V \to V'$ satisfying $\xi = \xi'(\varphi \wedge \varphi)$ and $\varphi\eta = \eta'(\varphi \wedge \varphi)$. If (V, ξ, η) is a dissident triple, then the vector space $\mathscr{F}(V, \xi, \eta) = \mathbb{R} \times V$, equipped with the multiplication

$$(\alpha, v)(\beta, w) = (\alpha\beta - \langle v, w \rangle + \xi(v \wedge w),\ \alpha w + \beta v + \eta(v \wedge w)),$$

is a real quadratic division algebra. If $\varphi : (V, \xi, \eta) \to (V', \xi', \eta')$ is a morphism of dissident triples, then the linear map

$$\mathscr{F}(\varphi) : \mathscr{F}(V, \xi, \eta) \to \mathscr{F}(V', \xi', \eta'),\ \mathscr{F}(\varphi)(\alpha, v) = (\alpha, \varphi(v))$$

is a morphism of real quadratic division algebras. It is well-known [5],[9] that Osborn's theorem [14] can be rephrased in the language of categories and functors as follows.

Theorem 3.1. *The functor $\mathscr{F} : \mathscr{V} \to \mathscr{D}^q$ is an equivalence of categories.*

For each $d \in \{1, 3, 5\}$ we denote by $\mathscr{V}_7^d$ the full subcategory of $\mathscr{V}$ formed by all dissident triples (V, ξ, η) satisfying $\dim(V) = 7$ and $\deg(\eta) = d$. Then the equivalence of categories $\mathscr{F} : \mathscr{V} \to \mathscr{D}^q$ induces equivalences of categories $\mathscr{F}_7^d : \mathscr{V}_7^d \to \mathscr{D}_8^{qd}$ for all $d \in \{1, 3, 5\}$. The category $\mathscr{V}_7^1$ admits an equivalent description entirely in terms of matrices, which we proceed to recall.

The octonion algebra $\mathbb{O}$ is well-known to be quadratic. Hence it has Frobenius decomposition $\mathbb{O} = \mathbb{R}1 \oplus V$, and thereby it determines $\mathbb{R}$-linear maps $\varrho : \mathbb{O} \to \mathbb{R}$ and $\iota : \mathbb{O} \to V$ such that $a = \varrho(a)1 + \iota(a)$ for all $a \in \mathbb{O}$. The symmetric $\mathbb{R}$-bilinear form

$$\mathbb{O} \times \mathbb{O} \to \mathbb{R},\ \langle x, y \rangle = 2\varrho(x)\varrho(y) - \frac{1}{2}\varrho(xy + yx)$$

is well-known to be positive definite, thus equipping $\mathbb{O}$ with the structure of a Euclidean space. Every algebra automorphism $\alpha \in \mathrm{Aut}(\mathbb{O})$ fixes the unity 1 of $\mathbb{O}$ and is orthogonal. Since $1^\perp = V$, it induces an orthogonal linear endomorphism $\alpha_V \in \mathrm{O}(V)$. The map $\nu : \mathrm{Aut}(\mathbb{O}) \to \mathrm{O}(V),\ \nu(\alpha) = \alpha_V$ is an injective group homomorphism. Choosing an orthonormal basis in V, the subgroup $\nu(\mathrm{Aut}(\mathbb{O})) < \mathrm{O}(V)$ is identified with a subgroup of $\mathrm{O}(7)$, which classically is denoted by $\mathbb{G}_2$. Simultaneously, the dissident map

$$\eta : V \wedge V \to V,\ \eta(v \wedge w) = \iota(vw)$$

is identified with a *vector product map* $\mathbb{R}^7 \wedge \mathbb{R}^7 \to \mathbb{R}^7,\ v \wedge w \mapsto v \times w$. Now denote by $\mathbb{R}^{7\times 7}$ the set of all real 7×7-matrices, and by $\mathbb{R}^{7\times 7}_{\mathrm{ant}}$, $\mathbb{R}^{7\times 7}_{\mathrm{pds}}$, $\mathbb{R}^{7\times 7}_{\mathrm{spds}}$ the subsets of $\mathbb{R}^{7\times 7}$ consisting of all matrices which are antisymmetric, positive definite symmetric, and positive definite symmetric of determinant 1 respectively. We view the matrix quadruple set

$$\mathscr{Q} = \mathbb{R}^{7\times 7}_{\mathrm{ant}} \times \mathbb{R}^{7\times 7}_{\mathrm{ant}} \times \mathbb{R}^{7\times 7}_{\mathrm{pds}} \times \mathbb{R}^{7\times 7}_{\mathrm{spds}}$$

as the object set of a groupoid $\mathscr{Q}$ whose morphisms

$$S : (A, B, C, D) \to (A', B', C', D')$$

are the orthogonal matrices $S \in \mathbb{G}_2$ which satisfy

$$(SAS^t, SBS^t, SCS^t, SDS^t) = (A', B', C', D').$$

Then the groupoid $\mathscr{Q}$ and the category $\mathscr{V}^1_7$ are related as follows. (For proofs see [7, Section 5].)

Theorem 3.2. *(i) If $(A, B, C, D) \in \mathscr{Q}$ then $\mathscr{G}(A, B, C, D) = \big(\mathbb{R}^7, \xi, \eta\big)$, with $\xi(v \wedge w) = v^t Aw$ and $\eta(v \wedge w) = (B + C)D(Dv \times Dw)$, is in $\mathscr{V}^1_7$.*

(ii) If $S : (A, B, C, D) \to (A', B', C', D')$ is a morphism in $\mathscr{Q}$ then $\mathscr{G}(S) : \mathscr{G}(A, B, C, D) \to \mathscr{G}(A', B', C', D')$, given by $\mathscr{G}(S)(v) = Sv$, is a morphism in $\mathscr{V}^1_7$.

(iii) The functor $\mathscr{G} : \mathscr{Q} \to \mathscr{V}^1_7$ is an equivalence of categories.

Composing the functors $\mathscr{G}$ and $\mathscr{F}^1_7$ to $\mathscr{H} = \mathscr{F}^1_7\mathscr{G}$, we arrive at the following explicit description of the category $\mathscr{D}^{q1}_8$ entirely in terms of matrices.

Corollary 3.3. *The functor $\mathscr{H} : \mathscr{Q} \to \mathscr{D}^{q1}_8$ is an equivalence of categories. It is given on objects by $\mathscr{H}(A, B, C, D) = \mathbb{R} \times \mathbb{R}^7$, with multiplication $(\alpha, v)(\beta, w) = (\alpha\beta - v^t w + v^t Aw, \alpha w + \beta v + (B + C)D(Dv \times Dw))$, and on morphisms by $\mathscr{H}(S)(\alpha, v) = (\alpha, Sv)$.*

On the other hand, we do not know any description of the categories $\mathscr{V}^3_7$ or $\mathscr{V}^5_7$ entirely in terms of matrices. Indeed, the 7-dimensional *dissident algebras* (V, η) of degree 3 or 5 which are inherent in the objects (V, ξ, η) of $\mathscr{V}^3_7$ or $\mathscr{V}^5_7$ respectively seem hardly to be understood at present.

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